

The Gardner method for symmetries

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January 12, 2013

Abstract

The Gardner method, traditionally used to generate conservation laws of integrable equations, is generalized to generate symmetries. The method is demonstrated for the KdV, Camassa-Holm and Sine-Gordon equations. The method involves identifying a symmetry which depends upon a parameter; expansion of this symmetry in a (formal) power series in the parameter then gives the usual infinite hierarchy of symmetries. We show that the obtained symmetries commute, discuss the relation of the Gardner method with Lenard recursion (both for generating symmetries and conservation laws), and also the connection between the symmetries of continuous integrable equations and their discrete analogs.

1 Introduction

More than forty years ago Miura discovered the so-called Miura map [18]. In a subsequent paper Miura, Gardner and Kruskal showed how to use the Miura map for the construction of an infinite hierarchy of conservation laws for the Korteweg-de Vries (KdV) equation [19]. For brevity we call this technique the Gardner method. Other techniques for the construction of conservation laws have appeared since then, for example Lenard recursion [9, 15, 16, 17, 23], the Gelfand-Dickey method [5, 10, 32] and the symmetry method [23]. Nevertheless the Gardner method is considered to be the simplest and most elegant method for the construction of conservation laws. It has been applied to other integrable equations, for instance to the Camassa-Holm(CH) equation [7].

An analog of the Gardner method for discrete equations appeared recently [25, 26]. An infinite number of conservation laws were constructed for discrete KdV (dKdV) and all the ABS equations [2] using Bäcklund transformations (BTs — note the Miura map is one of the defining equations for the BT of KdV). Moreover in [25] it was shown that it is also possible to use the Gardner method to construct the infinite hierarchy

of symmetries for dKdV. This result raises the question whether it is possible to do the same for continuous equations.

The theory of symmetries for continuous equations is well developed. There is a direct method which allows computation of all symmetries of a given order for a given equation. There are numerous methods to generate the infinite hierarchies of symmetries of integrable equations such as KdV, for example, the mastersymmetry method [8], Lenard recursion [22, 23, 24] and Lax operator methods [10, 15, 31]. The method we will present in this paper has much in common with the method of the resolvent, described, for KdV, in section 3.7 of [5].

The goal of this paper is to present the Gardner method for symmetries of continuous partial differential equations (PDE). It is a little more subtle than the Gardner method for conservation laws, but it is still simple and elegant. The structure of this paper is as follows: In section 2 the Gardner method for symmetries is presented for KdV, CH and sine-Gordon (SG) equations. In section 3 we prove that the obtained symmetries commute. In section 4 we show the connection of the Gardner method and Lenard recursion. In section 5 the connection between the symmetries of continuous and discrete equations is described. Section 6 contains some concluding comments and questions for further study.

2 Gardner method for symmetries

2.1 KdV

The potential KdV (pKdV) and KdV equations have the form

$$u_t - \frac{3}{2}u_x^2 - \frac{1}{4}u_{xxx} = 0, \quad (1)$$

$$\phi_t - 3\phi\phi_x - \frac{1}{4}\phi_{xxx} = 0 \quad (2)$$

related by $\phi = u_x$. The standard Bäcklund transformation for u is $u \rightarrow u_\alpha = u + v_\alpha$ where v_α satisfies

$$v_{\alpha,x} = \alpha - 2\phi - v_\alpha^2, \quad (3)$$

$$v_{\alpha,t} = -\frac{1}{2}\phi_{xx} + (\alpha + \phi)(\alpha - 2\phi - v_\alpha^2) + \phi_x v_\alpha. \quad (4)$$

The index α indicates that v_α is the solution of (3) with parameter α . The system (3)-(4) is consistent if ϕ satisfies the KdV equation (2) and has a one-parameter family of solutions. By $v_{\alpha_1}, v_{\alpha_2}$ we denote different solutions of (3)-(4) with the same parameter α . There is an algebraic way to describe the action of repeated Bäcklund transformations with different parameters [6]. This relation is called the nonlinear superposition principle and has the form

$$u_{\alpha,\beta} = u + \frac{\beta - \alpha}{u_\beta - u_\alpha}. \quad (5)$$

Here $u_\alpha = u + v_\alpha$ is obtained by application of the BT with parameter α , $u_\beta = u + v_\beta$ is from application of the BT with parameter β , and $u_{\alpha,\beta}$ is the result of application of both, in either order, as they commute. Writing $\beta = \epsilon + \alpha$, relation (5) can be rewritten

$$u_{\alpha,\alpha+\epsilon} = u + \frac{\epsilon}{u_{\alpha+\epsilon} - u_\alpha} = u + \frac{\epsilon}{v_{\alpha+\epsilon} - v_\alpha}. \quad (6)$$

For small ϵ we can expand $v_{\alpha+\epsilon}$ as a power series in ϵ . To lowest order, $v_{\alpha+\epsilon}$ satisfies the same system of differential equations as v_α . But this does *not* mean that to lowest order it is the same as v_α , since, as we have explained, the BT has a one-parameter family of solutions. Relation (6) can thus be rewritten in the form

$$u_{\alpha,\alpha+\epsilon} = u + \frac{\epsilon}{v_{\alpha_1} - v_{\alpha_2}} + O(\epsilon^2). \quad (7)$$

where v_{α_1} and v_{α_2} are distinct solutions of (3)-(4) for the same parameter value α .

Relation (7) describes an infinitesimal continuous transformation of u which is also a solution of pKdV. In other words, this is a Lie symmetry for pKdV. We denote it $X(\alpha) = Q(\alpha) \frac{\partial}{\partial u}$ where

$$Q(\alpha) = \frac{1}{v_{\alpha_1} - v_{\alpha_2}}. \quad (8)$$

We call $X(\alpha)$ *the general symmetry*. Note that while the explanation we have given above for why $X(\alpha)$ is a Lie symmetry is perfectly rigorous, there is a more direct technical proof: If Q is defined as in (8) and v_{α_1} and v_{α_2} are both solutions of (3)-(4) then it is a technical exercise to check that Q satisfies the equation

$$Q_t - 3\phi Q_x - \frac{1}{4}Q_{xxx} = 0, \quad (9)$$

this being the defining equation for infinitesimal symmetries of pKdV (1).

The next thing to do is to observe that if we could solve (3)-(4) to write v_α as a function of ϕ (or u) and (a finite number of) its derivatives, as well as the constant of integration, then we could rewrite (8) in terms of ϕ . This cannot be done explicitly, but it is possible to write a formal solution of (3)-(4) to give v_α in terms of ϕ as a formal series in decreasing powers of $\alpha^{1/2}$. Explicitly we have

$$v_\alpha = \alpha^{1/2} - \frac{\phi}{\alpha^{1/2}} + \frac{\phi_x}{2\alpha} - \frac{\phi_{xx} + 2\phi^2}{4\alpha^{3/2}} + \frac{\phi_{xxx} + 8\phi\phi_x}{8\alpha^2} - \frac{\phi_{xxxx} + 8\phi^3 + 10\phi_x^2 + 12\phi\phi_{xx}}{16\alpha^{5/2}} + O(\alpha^{-3}) \quad (10)$$

or

$$v_\alpha = \alpha^{1/2} + \sum_{i=1}^{\infty} \frac{c_i}{\alpha^{i/2}} \quad (11)$$

where

$$c_1 = -\phi, \quad c_2 = \frac{\phi_x}{2}, \quad c_{n+1} = -\frac{1}{2} \left(c_{n,x} + \sum_{i=1}^{n-1} c_i c_{n-i} \right), \quad n = 2, 3, \dots \quad (12)$$

At first glance this seems insufficient for our purpose, as it only gives a single solution of equations (3)-(4), with the prescribed asymptotic behavior for large $|\alpha|$. But there evidently is a second solution in which $\alpha^{1/2}$ is replaced by $-\alpha^{1/2}$. Thus we can take

$$v_{\alpha_1} = \alpha^{1/2} + \sum_{i=1}^{\infty} \frac{c_i}{\alpha^{i/2}}, \quad v_{\alpha_2} = -\alpha^{1/2} + \sum_{i=1}^{\infty} \frac{(-1)^i c_i}{\alpha^{i/2}}, \quad (13)$$

with the c_i defined as before to obtain

$$\begin{aligned} Q(\alpha) &= \frac{1}{2\alpha^{1/2}} \frac{1}{\left(1 + \sum_{j=0}^{\infty} \frac{c_{2j+1}}{\alpha^{j+1}}\right)} \\ &= \frac{1}{2\alpha^{1/2}} \left(1 + \frac{\phi}{\alpha} + \frac{\phi_{xx} + 6\phi^2}{4\alpha^2} + \frac{\phi_{xxxx} + 40\phi^3 + 10\phi_x^2 + 20\phi\phi_{xx}}{16\alpha^3} + \dots\right). \end{aligned} \quad (14)$$

This expansion gives the infinite hierarchy of symmetries of pKdV. The first few symmetries take the form

$$\begin{aligned} X_0 &= \frac{\partial}{\partial u} , \\ X_1 &= u_x \frac{\partial}{\partial u} , \\ X_2 &= (u_{xxx} + 6u_x^2) \frac{\partial}{\partial u} = 4u_t \frac{\partial}{\partial u} , \\ X_3 &= (u_{xxxxx} + 40u_x^3 + 10u_{xx}^2 + 20u_x u_{xxx}) \frac{\partial}{\partial u} . \end{aligned}$$

The corresponding symmetry for KdV is

$$Y(\alpha) = Q(\alpha)_x \frac{\partial}{\partial \phi} . \quad (15)$$

(Note that in general it is nontrivial that a local symmetry of pKdV should give a local symmetry of KdV, as u is nonlocal in ϕ , but since all the symmetries for pKdV we are considering are determined by ϕ this is not an issue here.) Using (8),(3) and (13) we have

$$Q(\alpha)_x = -\frac{v_{\alpha_1,x} - v_{\alpha_2,x}}{(v_{\alpha_1} - v_{\alpha_2})^2} = \frac{v_{\alpha_1} + v_{\alpha_2}}{v_{\alpha_1} - v_{\alpha_2}} = \frac{1}{\alpha^{3/2}} \frac{\left(\sum_{j=0}^{\infty} \frac{c_{2j+2}}{\alpha^j}\right)}{\left(1 + \sum_{j=0}^{\infty} \frac{c_{2j+1}}{\alpha^{j+1}}\right)} . \quad (16)$$

Expanding in powers of $\frac{1}{\alpha}$, or just differentiating the relevant formulas for pKdV, gives the first few symmetries for KdV

$$\begin{aligned} Y_1 &= \phi_x \frac{\partial}{\partial \phi} , \\ Y_2 &= (\phi_{xxx} + 12\phi\phi_x) \frac{\partial}{\partial \phi} = 4\phi_t \frac{\partial}{\partial \phi} , \\ Y_3 &= (\phi_{xxxxx} + 120\phi^2\phi_x + 40\phi_x\phi_{xx} + 20\phi\phi_{xxx}) \frac{\partial}{\partial \phi} . \end{aligned}$$

Thus we can generate explicit formulas for the infinite hierarchy of symmetries of pKdV or KdV using (12) to find the c_i and then expanding (14) for pKdV or (16) for KdV in inverse powers of α .

In fact the function $Q(\alpha)$ that has emerged as a generating function for symmetries of KdV can be identified with the “resolvent” of KdV (see [5] section 3.7). Writing $v_\alpha = \frac{\psi_{\alpha x}}{\psi_\alpha}$, equation (3) becomes the Schrödinger equation

$$\psi_{\alpha xx} = (\alpha - 2\phi)\psi_\alpha .$$

Using (8) and the fact that the Wronskian of two solutions of the Schrödinger equation is a constant, we find that $Q(\alpha)$ can be identified with the product of two solutions of the Schrödinger equation, which is the resolvent. However, the proof we have given above that $Q(\alpha)$ is a symmetry of pKdV is new, and as we shall see, this new proof allows generalization to other equations, as well as a simpler proof of properties such as commutativity.

2.2 CH

In [7] the conserved quantities of CH were derived from those of the associated Camassa-Holm (ACH), introduced in [29] and studied further in [12, 13, 14, 28]. The ACH equation has the form

$$p_t = p^2 f_x, \quad f = \frac{p}{4} \left(\frac{p_t}{p} \right)_x - \frac{p^2}{2}. \quad (17)$$

In [29] a BT for ACH was given in the form $p \rightarrow p - s_x$ where s satisfies

$$s_x = -\frac{s^2}{p\alpha} + \frac{\alpha}{p} + p, \quad (18)$$

$$s_t = -s^2 + \frac{p_t}{p}s + \alpha(\alpha - 2f). \quad (19)$$

The superposition principle for ACH is

$$p_{\alpha,\beta} = p - \left(\frac{(\alpha - \beta)(\alpha\beta - s_\alpha s_\beta)}{\beta s_\alpha - \alpha s_\beta} \right)_x. \quad (20)$$

Here $p_\alpha = p - s_{\alpha,x}$ is obtained by application of the BT with parameter α , $p_\beta = p - s_{\beta,x}$ is from application of the BT with parameter β , and $p_{\alpha,\beta}$ is the result of application of both. Writing $\beta = \epsilon + \alpha$, and doing the expansion of (20) around $\epsilon = 0$ we obtain

$$p_{\alpha,\alpha+\epsilon} = p - \epsilon \left(\frac{\alpha^2 - s_{\alpha_1} s_{\alpha_2}}{s_{\alpha_1} - s_{\alpha_2}} \right)_x + O(\epsilon^2) \quad (21)$$

where s_{α_1} and s_{α_2} are distinct solutions of (18,19) for the same parameter value α . Relation (21) describes an infinitesimal continuous transformation of p which is also a solution of ACH. We denote it $X(\alpha) = Q(\alpha) \frac{\partial}{\partial p}$ where

$$Q(\alpha) = \left(\frac{\alpha^2 - s_{\alpha_1} s_{\alpha_2}}{s_{\alpha_1} - s_{\alpha_2}} \right)_x = \frac{p(s_{\alpha_1} + s_{\alpha_2})}{s_{\alpha_1} - s_{\alpha_2}}. \quad (22)$$

Using (17,18,19) it can be checked that the coefficient of $\frac{\partial}{\partial f}$ in the prolongation of $X(\alpha)$ is

$$\tilde{Q} = \frac{\alpha(s_{\alpha_1} + s_{\alpha_2} - pf_x)}{s_{\alpha_1} - s_{\alpha_2}}. \quad (23)$$

Furthermore Q, \tilde{Q} satisfy the equation for infinitesimal symmetries of ACH

$$Q_t = 2pf_x Q + p^2 \tilde{Q}_x. \quad (24)$$

To generate the symmetries we apply the same idea as in the previous section. Namely, we write a formal solutions of (18,19) as a series in increasing powers $\alpha^{\frac{1}{2}}$

$$s_\alpha = \sum_{n=1}^{\infty} s_n \alpha^{\frac{n}{2}},$$

where

$$s_1 = p, \quad s_2 = -\frac{p_x}{2}, \quad s_{n+1} = -\frac{s_{n,x}}{2} + \frac{1}{2p} \left(\delta_{n2} - \sum_{i=0}^{n-2} s_{i+2} s_{n-i} \right), \quad n = 2, 3, \dots$$

The second solution of (18,19) can be obtained by replacing $\alpha^{\frac{1}{2}}$ by $-\alpha^{\frac{1}{2}}$. Thus we take

$$s_{\alpha_1} = \sum_{n=1}^{\infty} s_n \alpha^{\frac{n}{2}}, \quad s_{\alpha_2} = \sum_{n=1}^{\infty} s_n (-\alpha^{\frac{1}{2}})^n.$$

Plugging this into (22) we obtain

$$\frac{Q(\alpha)}{\sqrt{\alpha}} = \frac{p \sum_{n=1}^{\infty} s_{2n} \alpha^n}{\sum_{n=1}^{\infty} s_{2n-1} \alpha^n}. \quad (25)$$

The expansion of (25) around $\alpha = 0$ gives the infinite hierarchy of symmetries of ACH. The first few take the form

$$\begin{aligned} X_1 &= p_x \frac{\partial}{\partial p}, \\ X_2 &= \left(2p_{xxx} - 12 \frac{p_x}{p^2} - 6 \frac{p_x p_{xx}}{p} + 3 \frac{p_x^3}{p^2} \right) \frac{\partial}{\partial p}, \\ X_3 &= \left(\frac{2}{5} p_{xxxx} - 2 \frac{p_x p_{xxxx} + 2 p_{xx} p_{xxx}}{p} + \frac{11 p_x p_{xx}^2 + 7 p_x^2 p_{xxx} - 4 p_{xxx}}{p^2} \right. \\ &\quad \left. + \frac{p_x p_{xx} (28 - 19 p_x^2)}{p^3} + \frac{3 p_x (16 - 40 p_x^2 + 9 p_x^4)}{4 p^4} \right) \frac{\partial}{\partial p}. \end{aligned}$$

As far as we are aware these symmetries are new.

2.3 SG

The equation SG

$$u_{xy} = \sin(u)$$

can be brought to rational form by the change of variable $u = 2i \ln(z)$, giving

$$z z_{xy} - z_x z_y = \frac{1}{4} (z^4 - 1). \quad (26)$$

The BT for (26) [4] is $z \rightarrow z_\alpha$ where z_α satisfies

$$(z_\alpha z)_x = \frac{\alpha}{2} (z_\alpha^2 - z^2), \quad z z_{\alpha,y} - z_y z_\alpha = \frac{1}{2\alpha} (z^2 z_\alpha^2 - 1). \quad (27)$$

The corresponding superposition principle is [11]

$$z_{\alpha,\beta} = z - (\alpha + \beta) \frac{z(z_\alpha - z_\beta)}{\beta z_\alpha - \alpha z_\beta} \quad (28)$$

Plugging $\beta = -\alpha + \epsilon$ and expanding (20) around $\epsilon = 0$ we obtain

$$z_{\alpha, -\alpha + \epsilon} = z + \epsilon \frac{z(z_{\alpha_1} + z_{\alpha_2})}{\alpha(z_{\alpha_1} - z_{\alpha_2})} + O(\epsilon^2), \quad (29)$$

where $z_{\alpha_1} = z_\alpha$ and $z_{\alpha_2} = -z_{-\alpha}$ are distinct solutions of (27) for the same parameter value α . Relation (29) describes an infinitesimal continuous transformation of z which is also a solution of SG. We denote it $X(\alpha) = Q(\alpha) \frac{\partial}{\partial z}$ where

$$Q(\alpha) = \frac{z(z_{\alpha_1} + z_{\alpha_2})}{(z_{\alpha_1} - z_{\alpha_2})}. \quad (30)$$

One can check that Q satisfies the equation for infinitesimal symmetries of SG

$$Qz_{xy} + zQ_{xy} - Q_xz_y - z_xQ_y - z^3Q = 0.$$

We write a formal solution of (27) as a series in decreasing powers of α

$$z_\alpha = \sum_{n=0}^{\infty} \frac{v_n}{\alpha^n},$$

where

$$v_0 = \pm z, \quad v_1 = \pm 2zz_x, \quad v_{n+1} = \frac{z_xv_n + zv_{n,x} - \frac{1}{2}\sum_{i=1}^n v_iv_{n+1-i}}{v_0}, \quad n = 2, 3, \dots$$

There are two distinct expansions of z_α . They can be used as two different solutions z_{α_1} and z_{α_2} of (27). Plugging z_{α_1} and z_{α_2} into (30) and expanding around $\alpha = 0$ we obtain an infinite hierarchy of symmetries. The first few symmetries take the form

$$\begin{aligned} X_1 &= z_x \frac{\partial}{\partial z}, \\ X_2 &= \left(z_{xxx} - 3 \frac{z_x z_{xx}}{z} \right) \frac{\partial}{\partial z}, \\ X_3 &= \left(z_{xxxx} + \frac{20z_x z_{xx}^2}{z^2} + \frac{10z_x^2 z_{xxx}}{z^2} - \frac{10z_{xx} z_{xxx}}{z} - \frac{10z_x^3 z_{xx}}{z^3} - \frac{5z_x z_{xxxx}}{z} \right) \frac{\partial}{\partial z}. \end{aligned}$$

These symmetries coincide with ones obtained in [22]. However, as far as we know the derivation above is new and represents a substantial simplification over existing techniques to generate the hierarchy. It is straightforward to verify that the j 'th symmetry for all equations (in the above numbering) depends on x -derivatives of u, p, z up to order $2j - 1$, and that the coefficient of the highest derivative is nonzero, thus guaranteeing nontriviality.

3 Algebra of the symmetries

In this section we determine the algebra of the symmetries obtained in the previous section. We prove that symmetries generated by the Gardner method for KdV, CH and SG commute. In the previous section we showed that symmetries for KdV can be obtained by the expansion of the general symmetry $X(\alpha)$ in a series around $\alpha = \infty$. In order to prove that these symmetries commute it is enough to show the general symmetry commutes with itself but with a different parameter, namely that

$$[X(\alpha), X(\beta)] = 0. \quad (31)$$

(We refer in this section only to the symmetry for pKdV, the result follows for the general symmetry of KdV $Y(\alpha)$ by a simple prolongation argument.) Here

$$X(\alpha) = \frac{1}{v_{\alpha_1} - v_{\alpha_2}} \frac{\partial}{\partial u}, \quad X(\beta) = \frac{1}{v_{\beta_1} - v_{\beta_2}} \frac{\partial}{\partial u}. \quad (32)$$

To compute the commutator we need to know how $X(\alpha)$ acts on v_{β_1} and v_{β_2} and how $X(\beta)$ acts on v_{α_1} and v_{α_2} . This is currently not clear. However we can also write $X(\alpha), X(\beta)$ in the form

$$X(\alpha) = \frac{1}{u_{\alpha_1} - u_{\alpha_2}} \frac{\partial}{\partial u}, \quad X(\beta) = \frac{1}{u_{\beta_1} - u_{\beta_2}} \frac{\partial}{\partial u}. \quad (33)$$

The relevant prolongations of $X(\alpha)$ and $X(\beta)$ are then

$$\hat{X}(\alpha) = \frac{1}{u_{\alpha_1} - u_{\alpha_2}} \frac{\partial}{\partial u} + \frac{1}{u_{\alpha_1\beta_1} - u_{\alpha_2\beta_1}} \frac{\partial}{\partial u_{\beta_1}} + \frac{1}{u_{\alpha_1\beta_2} - u_{\alpha_2\beta_2}} \frac{\partial}{\partial u_{\beta_2}}, \quad (34)$$

$$\hat{X}(\beta) = \frac{1}{u_{\beta_1} - u_{\beta_2}} \frac{\partial}{\partial u} + \frac{1}{u_{\alpha_1\beta_1} - u_{\alpha_1\beta_2}} \frac{\partial}{\partial u_{\alpha_1}} + \frac{1}{u_{\alpha_2\beta_1} - u_{\alpha_2\beta_2}} \frac{\partial}{\partial u_{\alpha_2}}. \quad (35)$$

To determine the commutator of $X(\alpha)$ and $X(\beta)$ we compute

$$\begin{aligned} \hat{X}(\alpha)Q(\beta) - \hat{X}(\beta)Q(\alpha) &= \hat{X}(\alpha) \left(\frac{1}{u_{\beta_1} - u_{\beta_2}} \right) - \hat{X}(\beta) \left(\frac{1}{u_{\alpha_1} - u_{\alpha_2}} \right) \\ &= \frac{1}{(u_{\beta_1} - u_{\beta_2})^2} \left(-\frac{1}{u_{\alpha_1\beta_1} - u_{\alpha_2\beta_1}} + \frac{1}{u_{\alpha_1\beta_2} - u_{\alpha_2\beta_2}} \right) \\ &\quad - \frac{1}{(u_{\alpha_1} - u_{\alpha_2})^2} \left(-\frac{1}{u_{\alpha_1\beta_1} - u_{\alpha_1\beta_2}} + \frac{1}{u_{\alpha_2\beta_1} - u_{\alpha_2\beta_2}} \right). \end{aligned} \quad (36)$$

In order to simplify this expression we have to use relation (5). Since we have two different BTs of u for parameter α and two different BTs of u for parameter β we need relation (5) in the four different forms

$$u_{\alpha_i\beta_j} = u + \frac{\alpha - \beta}{u_{\alpha_i} - u_{\beta_j}}, \quad i, j \in \{1, 2\}. \quad (37)$$

With the help of (37) the right hand side of (36) simplifies to zero, thereby establishing (31). Thus the symmetries obtained by the Gardner method for pKdV (and KdV) generate an abelian group.

Let us prove a similar result for ACH. For this we need to compute the prolongation of the infinitesimal generator of the general symmetry. This can be done by the change of variable

$$u = \int p dx. \quad (38)$$

The superposition principle for ACH can be rewritten in the form

$$u_{\alpha,\beta} = u - \left(\frac{(\alpha - \beta)(\alpha\beta - (u - u_\alpha)(u - u_\beta))}{\beta(u - u_\alpha) - \alpha(u - u_\beta)} \right). \quad (39)$$

This expression is equivalent to quad-graph equation $Q1_{\delta=1}$ in the ABS classification [2]. Using (38), the general symmetry of ACH is

$$X(\alpha) = \frac{\alpha^2 - (u - u_{\alpha_1})(u - u_{\alpha_2})}{u_{\alpha_2} - u_{\alpha_1}} \frac{\partial}{\partial u}.$$

In order to prove that the symmetries commute it is enough to show that (31) holds for the general symmetry of ACH. The relevant prolongations of $X(\alpha)$ and $X(\beta)$ are

$$\begin{aligned} \hat{X}(\alpha) &= \frac{\alpha^2 - (u - u_{\alpha_1})(u - u_{\alpha_2})}{u_{\alpha_2} - u_{\alpha_1}} \frac{\partial}{\partial u} + \frac{\alpha^2 - (u_{\beta_1} - u_{\beta_1\alpha_1})(u_{\beta_1} - u_{\beta_1\alpha_2})}{u_{\beta_1\alpha_2} - u_{\beta_1\alpha_1}} \frac{\partial}{\partial u_{\beta_2}} \\ &\quad + \frac{\alpha^2 - (u_{\beta_2} - u_{\beta_2\alpha_1})(u_{\beta_2} - u_{\beta_2\alpha_2})}{u_{\beta_2\alpha_2} - u_{\beta_2\alpha_1}} \frac{\partial}{\partial u_{\beta_2}}, \\ \hat{X}(\beta) &= \frac{\beta^2 - (u - u_{\beta_1})(u - u_{\beta_2})}{u_{\beta_2} - u_{\beta_1}} \frac{\partial}{\partial u} + \frac{\beta^2 - (u_{\alpha_1} - u_{\beta_1\alpha_1})(u_{\alpha_1} - u_{\beta_2\alpha_1})}{u_{\beta_2\alpha_1} - u_{\beta_1\alpha_1}} \frac{\partial}{\partial u_{\alpha_1}} \\ &\quad + \frac{\beta^2 - (u_{\alpha_2} - u_{\beta_1\alpha_2})(u_{\alpha_2} - u_{\beta_2\alpha_2})}{u_{\beta_2\alpha_2} - u_{\beta_1\alpha_2}} \frac{\partial}{\partial u_{\alpha_2}}. \end{aligned}$$

Relation (39) can be presented in four different forms which connect different BTs for CH

$$u_{\alpha_i, \beta_j} = u - \left(\frac{(\alpha_i - \beta_j)(\alpha\beta - (u - u_{\alpha_i})(u - u_{\beta_j}))}{\beta(u - u_{\alpha_i}) - \alpha(u - u_{\beta_j})} \right), \quad i, j \in \{1, 2\}. \quad (40)$$

With the help of this we obtain that (31) is true for the general symmetry of ACH. Thus the symmetries obtained by the Gardner method for ACH generate an abelian group.

The proof that symmetries obtained by the Gardner method for SG commute is very similar to the above. The prolongations of the general symmetries $X(\alpha)$ and $X(\beta)$ for SG are

$$\begin{aligned} \widehat{X}(\alpha) &= \frac{z(z_{\alpha_1} + z_{\alpha_2})}{(z_{\alpha_1} - z_{\alpha_2})} \frac{\partial}{\partial z} + \frac{z_{\beta_1}(z_{\beta_1\alpha_1} + z_{\beta_1\alpha_2})}{(z_{\beta_1\alpha_1} - z_{\beta_1\alpha_2})} \frac{\partial}{\partial z_{\beta_1}} + \frac{z_{\beta_2}(z_{\beta_2\alpha_1} + z_{\beta_2\alpha_2})}{(z_{\beta_2\alpha_1} - z_{\beta_2\alpha_2})} \frac{\partial}{\partial z_{\beta_2}}, \\ \widehat{X}(\beta) &= \frac{z(z_{\beta_1} + z_{\beta_2})}{(z_{\beta_1} - z_{\beta_2})} \frac{\partial}{\partial z} + \frac{z_{\alpha_1}(z_{\beta_1\alpha_1} + z_{\beta_2\alpha_1})}{(z_{\beta_1\alpha_1} - z_{\beta_2\alpha_1})} \frac{\partial}{\partial z_{\alpha_1}} + \frac{z_{\alpha_2}(z_{\beta_1\alpha_2} + z_{\beta_2\alpha_2})}{(z_{\beta_1\alpha_2} - z_{\beta_2\alpha_2})} \frac{\partial}{\partial z_{\alpha_2}}. \end{aligned}$$

Relations which connect different BTs for SG can be obtained from (28)

$$z_{\alpha_i, \beta_j} = z - (\alpha + \beta) \frac{z(z_{\alpha_i} - z_{\beta_j})}{\beta z_{\alpha_i} - \alpha z_{\beta_j}}, \quad i, j \in \{1, 2\}. \quad (41)$$

This helps us to obtain that (31) is true for the general symmetry of SG. Thus the symmetries obtained by the Gardner method for SG also generate an abelian group.

4 Connection of Gardner method with Lenard recursion

As we mentioned in the introduction there are several methods for generation of symmetries for PDE. One of them is Lenard recursion [22, 23, 24]. We intend to show that Lenard recursion gives the result equivalent to the Gardner method. We present a brief explanation of Lenard recursion for the KdV equation.

The Lenard recursion is based on the fact that KdV (2) is a bi-Hamiltonian system. Namely KdV can be presented in two forms:

$$u_t = P_0 \frac{\delta H_0}{\delta \phi}, \quad (42)$$

$$u_t = P_1 \frac{\delta H_1}{\delta \phi}, \quad (43)$$

where H_0 and H_1 are

$$H_0 = \int \left(\frac{\phi^3}{2} - \frac{\phi_x^2}{8} \right) dx, \quad H_1 = \int \frac{\phi^2}{2} dx,$$

and

$$P_0 = \frac{\partial}{\partial x}, \quad P_1 = \frac{1}{4} \frac{\partial^3}{\partial x^3} + 2\phi \frac{\partial}{\partial x} + \phi_x.$$

The expression $\frac{\delta}{\delta \phi}$ denotes the variational derivative. If $g = \int G dx$ is a conserved quantity for KdV equation then both $Q_0 = P_0 \frac{\delta g}{\delta \phi}$, $Q_1 = P_1 \frac{\delta g}{\delta \phi}$ are the characteristics of symmetries.

Evidentiary

$$P_1 P_0^{-1} Q_0 = Q_1.$$

Thus the 'recursion operator' for symmetries

$$R = P_1 P_0^{-1}$$

maps Hamiltonian symmetries to Hamiltonian symmetries, and can be used to generate the infinite hierarchy of symmetries of KdV. Though it is necessary to check that at each step the operator P_0^{-1} can be applied.

In section 2 we showed that KdV has a symmetry with characteristic

$$Q(\alpha)_x = \left(\frac{1}{v_{\alpha_1} - v_{\alpha_2}} \right)_x = \frac{v_{\alpha_1} + v_{\alpha_2}}{v_{\alpha_1} - v_{\alpha_2}}.$$

Applying R to $Q(\alpha)_x$ we obtain

$$RQ(\alpha)_x = P_1 P_0^{-1} \left(\frac{1}{v_{\alpha_1} - v_{\alpha_2}} \right)_x = P_1 \frac{1}{v_{\alpha_1} - v_{\alpha_2}} = \frac{\alpha(v_{\alpha_1} + v_{\alpha_2})}{v_{\alpha_1} - v_{\alpha_2}} = \alpha Q(\alpha)_x. \quad (44)$$

Writing

$$Q(\alpha)_x = \frac{1}{\alpha^{1/2}} \sum_{i=0}^{\infty} \frac{Q_i}{\alpha^i}$$

and substituting into (44) we obtain

$$R \left(\sum_{i=0}^{\infty} \frac{Q_i}{\alpha^i} \right) = \sum_{i=0}^{\infty} \frac{Q_i}{\alpha^{i-1}}.$$

Comparing coefficients of powers of α we get

$$RQ_i = Q_{i+1}, \quad i = 0, 1, 2, \dots$$

Thus Lenard recursion gives the same symmetries as the Gardner method for KdV. Further more, in our approach we see immediately that all the characteristics are x derivatives.

We briefly demonstrate that Lenard recursion for conservation laws can be derived from the Gardner method. In the Gardner method the infinite hierarchy of conservation laws for KdV is obtained by expanding the conservation law

$$\partial_t(v_{\alpha_1} - v_{\alpha_2}) + \partial_x(-(u + \theta)(v_{\alpha_1} - v_{\alpha_2})) = 0. \quad (45)$$

into a series in α with the help of (13) [19]. The conserved quantity associated with (45) has the form $g = \int (v_{\alpha_1} - v_{\alpha_2}) dx$. We now prove that the variational derivative of g is

$$\frac{\delta g}{\delta \phi} = \frac{4}{v_{\alpha_1} - v_{\alpha_2}}. \quad (46)$$

The proof for (46) is as follows. We know from (3) that

$$v_{\alpha_1, x} + v_{\alpha_2, x} - 2\alpha + v_{\alpha_1}^2 + v_{\alpha_2}^2 = -4\phi$$

The variation of this expression is

$$\delta v_{\alpha_1, x} + \delta v_{\alpha_2, x} + 2v_{\alpha_1} \delta v_{\alpha_1} + 2v_{\alpha_2} \delta v_{\alpha_2} = -4\delta\phi.$$

Dividing this by $v_{\alpha_1} - v_{\alpha_2}$ we obtain

$$\frac{\delta v_{\alpha_1, x} + \delta v_{\alpha_2, x}}{v_{\alpha_1} - v_{\alpha_2}} + \frac{2v_{\alpha_1} \delta v_{\alpha_1} + 2v_{\alpha_2} \delta v_{\alpha_2}}{v_{\alpha_1} - v_{\alpha_2}} = -4 \frac{\delta\phi}{v_{\alpha_1} - v_{\alpha_2}}.$$

This expression can be rewritten differently as

$$-\frac{(\delta v_{\alpha_1} + \delta v_{\alpha_2})(v_{\alpha_1} + v_{\alpha_2})}{v_{\alpha_1} - v_{\alpha_2}} + \partial_x \left(\frac{\delta v_{\alpha_1} + \delta v_{\alpha_2}}{v_{\alpha_1} - v_{\alpha_2}} \right) + \frac{2v_{\alpha_1} \delta v_{\alpha_1} + 2v_{\alpha_2} \delta v_{\alpha_2}}{v_{\alpha_1} - v_{\alpha_2}} = -4 \frac{\delta\phi}{v_{\alpha_1} - v_{\alpha_2}}.$$

After simplification we obtain

$$\delta v_{\alpha_1} - \delta v_{\alpha_2} = 4 \frac{\delta\phi}{v_{\alpha_1} - v_{\alpha_2}} + \partial_x \left(\frac{\delta v_{\alpha_1} + \delta v_{\alpha_2}}{v_{\alpha_1} - v_{\alpha_2}} \right).$$

Thus

$$\delta \int (v_{\alpha_1} - v_{\alpha_2}) dx = \int \frac{4}{v_{\alpha_1} - v_{\alpha_2}} \delta\phi dx,$$

and (46) is proved. We have already seen in (44) that

$$P_1 \frac{1}{v_{\alpha_1} - v_{\alpha_2}} = \alpha Q(\alpha)_x = \alpha P_0 \frac{1}{v_{\alpha_1} - v_{\alpha_2}}.$$

Thus

$$P_1 \frac{\delta g}{\delta\phi} = \alpha P_0 \frac{\delta g}{\delta\phi},$$

or

$$\bar{R} \frac{\delta g}{\delta\phi} = \alpha \frac{\delta g}{\delta\phi},$$

where $\bar{R} = P_0^{-1} P_1$ is the recursion operator for (variational derivatives of) conserved quantities of KdV.

5 Connection with symmetries of difference equations.

Relation (5)

$$u_{\alpha\beta} = u + \frac{\alpha - \beta}{u_{\alpha} - u_{\beta}}$$

connects solutions of the pKdV equation. It can be embedded on the quad-graph, the planar graph with quadrilateral faces [2, 3, 20, 21, 30]. The embedded equation is called the discrete Korteweg-de Vries equation (dKdV) and has the following form

$$u_{1,1} = u_{0,0} + \frac{\alpha - \beta}{u_{1,0} - u_{0,1}}. \quad (47)$$

Here $k, l \in \mathbb{Z}^2$ are independent variables and $u_{0,0} = u(k, l)$ is a dependent variable that is defined on the domain \mathbb{Z}^2 . We denote the values of this variable on other points by

$u_{i,j} = u(k+i, l+j) = S_k^i S_l^j u_{0,0}$, where S_k, S_l are unit forward shift operators in k and l respectively. Let us explain the connection between (47) and (5). Shifts $u_{1,0}$ and $u_{0,1}$ are BTs of $u = u_{0,0}$ for two different parameters α, β . In general $u_{i,j}$ denotes the application of the BT $i+j$ times to $u = u_{0,0}$, i times with parameter α and j times with parameter β .

Properties of dKdV are closely related to properties of pKdV. In particular, since the continuous general symmetry of pKdV discussed in the previous sections is an infinitesimal version of a Bäcklund transformation, and Bäcklund transformations commute, the general symmetry must also give a continuous symmetry of dKdV, and we wish to identify its generator.

So far we have identified the functions $u_{\alpha_1}, u_{\alpha_2}$ in the pKdV general symmetry

$$X(\alpha) = \frac{1}{u_{\alpha_1} - u_{\alpha_2}} \frac{\partial}{\partial u} \quad (48)$$

as different Bäcklund transformations of u with the same parameter α . In fact we can just as well identify one of them as a forward Bäcklund transformation and another as a reverse Bäcklund transformation. To demonstrate this, observe that a reverse Bäcklund transformation of \tilde{u} is a function u such that $u = \tilde{u} + \tilde{v}$ where \tilde{v} solves the system

$$\tilde{v}_x = \alpha - 2\tilde{\phi} - \tilde{v}^2, \quad (49)$$

$$\tilde{v}_t = -\frac{1}{2}\tilde{\phi}_{xx} + (\alpha + \tilde{\phi})(\alpha - 2\tilde{\phi} - \tilde{v}^2) + \tilde{\phi}_x \tilde{v}, \quad (50)$$

and $\tilde{\phi} = \tilde{u}_x$. Replacing $\tilde{\phi}$ in (49)-(50) by $\phi - \tilde{v}_x$, and using the first equation in the second to eliminate all x derivatives of \tilde{v} , the above system becomes

$$-\tilde{v}_x = \alpha - 2\phi - \tilde{v}^2, \quad (51)$$

$$-\tilde{v}_t = -\frac{1}{2}\phi_{xx} + (\alpha + \phi)(\alpha - 2\phi - \tilde{v}^2) - \phi_x \tilde{v}. \quad (52)$$

Comparing with (3)-(4), we can identify \tilde{v} with $-v_\alpha$, where v_α is the function appearing in the forward Bäcklund transformation. In other words, *forward and reverse Bäcklund transformations are identical*.

In the language of the quad-graph, inverse shifts $u_{-1,0}$ and $u_{0,-1}$ are reverse BTs. So we can identify u_{α_2} with $u_{-1,0}$ and u_{α_1} with $u_{1,0}$ and the quad-graph version of $X(\alpha)$ is thus

$$X_1 = \frac{1}{u_{1,0} - u_{-1,0}} \frac{\partial}{\partial u_{0,0}}. \quad (53)$$

Similarly the quad-graph version of $X(\beta)$ is

$$X_2 = \frac{1}{u_{0,1} - u_{0,-1}} \frac{\partial}{\partial u_{0,0}}. \quad (54)$$

These symmetries were already found in [27]. Furthermore, in [26] it was shown that equation (47) embedded in three dimensions has another symmetry of the form

$$X = \frac{1}{u_{0,0,1} - u_{0,0,-1}} \frac{\partial}{\partial u_{0,0,0}}. \quad (55)$$

This evidently also has its origins in the general symmetry. In [26] it was shown how, by a suitable expansion, this symmetry gives the infinite hierarchy of symmetries for the dKdV equation.

As we noted before the CH equation after the change of variable $u = \int p dx$ is related to quad-graph equation $Q1_{\delta=1}$. Namely, the superposition principle for CH (39) is equivalent to quad-graph equation $Q1_{\delta=1}$ in the ABS classification [2]

$$\theta_2(u_{1,1} - u_{0,1})(u_{0,0} - u_{1,0}) - \theta_1(u_{1,1} - u_{1,0})(u_{0,0} - u_{0,1}) + (\theta_1 - \theta_2)\theta_1\theta_2 = 0.$$

So from the general symmetry of ACH we can obtain the symmetry of $Q1_{\delta=1}$.

By doing the same computation as we did for pKdV we can identify u_{α_1} as a forward Bäcklund transformation and u_{α_2} as a reverse Bäcklund transformation. Thus

$$X_1 = \left(\frac{\alpha^2 - (u_{0,0} - u_{1,0})(u_{0,0} - u_{-1,0})}{u_{-1,0} - u_{1,0}} \right) \frac{\partial}{\partial u_{0,0}},$$

$$X_2 = \left(\frac{\alpha^2 - (u_{0,0} - u_{0,1})(u_{0,0} - u_{0,-1})}{u_{0,-1} - u_{0,1}} \right) \frac{\partial}{\partial u_{0,0}}.$$

are the symmetries of $Q1_{\delta=1}$.

For SG we can identify z_{α_1} as a forward Bäcklund transformation and $-z_{\alpha_2}$ as a reverse Bäcklund transformation. From general symmetry of SG we obtain that

$$X_1 = \frac{z_{0,0}(z_{1,0} - z_{-1,0})}{(z_{1,0} + z_{-1,0})} \frac{\partial}{\partial z_{0,0}},$$

$$X_2 = \frac{z_{0,0}(z_{0,1} - z_{0,-1})}{(z_{0,1} + z_{0,-1})} \frac{\partial}{\partial z_{0,0}}.$$

are the symmetries for equation

$$\alpha(z_{0,0}z_{0,1} - z_{1,0}z_{1,1}) - \beta(z_{0,0}z_{1,0} - z_{0,1}z_{1,1}) = 0.$$

This equation is equivalent to $H3_{\delta=0}$ in the ABS classification.

6 Conclusion

In this paper we have introduced the Gardner method for generation of the infinite hierarchy of symmetries of integrable equations, using KdV, CH and SG as examples. The method involves identifying the general symmetry $X(\alpha)$ from the superposition principle for BTs of the equations studied, followed by a suitable expansion in powers of α . The method is both mathematically elegant and computationally efficient. We have shown how to use our formalism to prove the symmetries commute, explained the origin of Lenard recursion relations, and explored the link with integrable lattice equations. The fact that integrable lattice equations arise as the superposition principle for BTs of continuum equations is well-known — indeed the celebrated $Q4$ lattice equation in the ABS classification was first written down by Adler [1] as the superposition principle for the Krichever-Novikov equation. However, we believe the link between the $Q1_{\delta=1}$ lattice equation and CH given in this paper is new, as are the symmetries derived for the ACH equation.

We expect the method to be applicable to other integrable equations, and it will be interesting to see more examples developed. It would be particularly interesting to see examples in which infinite hierarchies of symmetries and/or conservation laws can be constructed by one method, but not by another (though this may be difficult to verify).

Acknowledgments We thank Peter Hydon for encouraging this line of research and Qiming Liu for significant comments.

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